On Dual Formulation of Gravity

Yu. M. Zinoviev *
Institute for High Energy Physics
Protvino, Moscow Region, 142280,Russia

Abstract

In this paper we consider a possibility to construct dual formulation of gravity where the main dynamical field is the Lorentz connection ${\omega_\mu}^{ab}$ and not that of tetrad ${e_\mu}^a$ or metric $g_{\mu\nu}$. Our approach is based on the usual dualization procedure which uses first order parent Lagrangians but in (Anti) de Sitter space and not in the flat Minkowski one. It turns out that in $d = 3$ dimensions such dual formulation is related with the so called exotic parity-violating interactions for massless spin-2 particles.

*E-mail address: yurii.zinoviev@ihep.ru

Introduction

Investigations of dual formulations for tensor fields are important for understanding of alternative formulations of known theories like gravity as well as understanding of their role in superstrings. Common procedure for obtaining such dual formulations is based on the parent first order Lagrangians. As is well known in flat Minkowski space such dualization procedure leads to different results for massive and massless particles. At the same time in (Anti) de Sitter space-time gauge invariance requires introduction quadratic mass-like terms into the Lagrangians. As a result dualization for massless particles in (Anti) de Sitter spaces [1] goes exactly in the same way as that for massive particles [2] and gives results different from ones for dualization of massless particles [3] in flat Minkowski space.

In this paper using such dualization procedure we consider a possibility to construct dual formulation of gravity where the main dynamical quantity is a Lorentz connection field ${\omega_\mu}^{ab}$. Note that at the Hamiltonian level such description was discussed in [4]. Also such dual formulation of $d = 3$ gravity was recently discussed in [5]. It turns out that in $d = 3$ dimensions such dual formulation of gravity is related with the so called exotic parity-violating interactions for massless spin-2 particles [6, 7]. So we start with $d = 3$ case and show that such exotic interaction can be viewed as higher derivatives interactions in terms Lorentz connection ${\omega_\mu}^{ab}$. Then we show how such interaction could be obtained from the usual gravitational interactions by dualization procedure starting with (Anti) de Sitter space and then considering a kind of flat limit. Then in the next section we consider straightforward generalization of such theory on arbitrary $d \geq 4$ dimensions.

1 Dual gravity in $d = 3$

Investigations of possible interactions for massless spin-2 particles have shown that in $d = 3$ case there exist non-trivial "exotic parity-violating" higher derivatives interactions [6, 7]. The simplest way to see this [8] is to start with the first order formulation for massless spin-2 particle using "triad" ${h_\mu}^a$ and Lorentz connection ${\omega_\mu}^{ab}$ and introduce dual variable ${f_\mu}^a = \frac{1}{2}\varepsilon^{abc}{\omega_\mu}^{bc}$. In this notations the Lagrangian for free massless spin-2 particles has a very simple form:

$$\mathcal{L}_0 = \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} {f_\mu}^a {f_\nu}^b - \varepsilon^{\mu\nu\alpha} {f_\mu}^a \partial_\nu {h_\alpha}^a \qquad (1)$$

Here

$$\left\{ {}^{\mu\nu}_{ab} \right\} = {\delta_a}^\mu {\delta_b}^\nu - {\delta_a}^\nu {\delta_b}^\mu$$

and so on. This Lagrangian is invariant under the following local gauge transformations:

$$\delta h_{\mu a} = \partial_\mu \xi_a + \varepsilon_{\mu ab}\eta^b \qquad \delta {f_\mu}^a = \partial_\mu \eta_a \qquad (2)$$

Then it is easy to check that if we add the following cubic terms to the Lagrangian and appropriate corrections to gauge transformation laws:

$$\mathcal{L}_1 = -\frac{K}{6}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} {f_\mu}^a {f_\nu}^b {f_\alpha}^c \qquad \delta_1 {h_\mu}^a = -K\varepsilon^{abc}{f_\mu}^b\eta^c \qquad (3)$$

where K — arbitrary coupling constant, we obtain gauge invariant interacting theory. In this, equation of motion for the $f_\mu{}^a$ field are still algebraic, but non-linear now. So if we try to solve this equation in passing to second order formulation we get essentially non-linear theory with higher and higher derivatives terms. To see what kind of theory we get let us consider lowest order approximations. It will be convenient to introduce "dual torsion"

$$T^{\mu a} = -\varepsilon^{\mu\nu\alpha}\partial_\nu h_\alpha{}^a, \qquad \hat{T}^{\mu a} = T^{\mu a} - e^{\mu a}T$$

Then from the quadratic Lagrangian we easily obtain:

$$f_\mu^{(1)a} = \hat{T}^a{}_\mu, \qquad \mathcal{L}_0 = -\frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \hat{T}^a{}_\mu \hat{T}^b{}_\nu$$

In the next quadratic order we get:

$$f_\mu^{(2)a} = -K\hat{T}^a{}_b\hat{T}^b{}_\mu + \frac{K}{4}e_\mu{}^a[\hat{T}^b{}_c\hat{T}^c{}_b + \hat{T}^2]$$

Substituting this expressions back to the first order Lagrangian and keeping only terms cubic in fields we obtain an interactions in a first non-trivial order:

$$\mathcal{L}_1 = \frac{K}{6}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{T}^a{}_\mu \hat{T}^b{}_\nu \hat{T}^c{}_\alpha \tag{4}$$

and this is just the interaction considered in [6, 7]. Note here that such interactions do not necessarily violate parity because one can always assign $f_\mu{}^a$ to be a tensor, while $h_\mu{}^a$ — a pseudotensor. Now we can once again use a peculiarity of $d = 3$ space and dualize $h_\mu{}^a$ instead of $f_\mu{}^a$: $h_\alpha{}^a = \frac{1}{2}\varepsilon^{abc}\omega_\alpha{}^{bc}$ Then we can rewrite all results in terms of this new variable by noting that:

$$\hat{T}^a{}_\mu = (R_\mu{}^a - \frac{1}{4}\delta_\mu{}^a R) = \hat{R}_\mu{}^a$$

where we have introduced usual field strength:

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab}, \qquad R_\mu{}^a = \delta^\nu{}_b R_{\mu\nu}{}^{ab}$$

Now the cubic interactions looks like:

$$\mathcal{L}_1 = \frac{K}{6}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{R}_\mu{}^a \hat{R}_\nu{}^b \hat{R}_\alpha{}^c \tag{5}$$

In $d = 3$ dimensions all these look just like trivial field redefinition, but looking this way it has to be clear that there should exist a generalization of such interactions on arbitrary $d \geq 4$. To see how this generalization could be constructed we have to reobtain the same results without use of peculiarities of $d = 3$ dimensions. Now we will show that it is indeed possible by following usual dualization procedure based on the parent first order Lagrangians. Crucial fact here is that dualization for massless particles in (Anti) de Sitter spaces [1] goes in way similar to the one for massive particles in flat space [2] and not to that for massless ones [3]. So let us return back to the free case and start with massless particle in (Anti) de Sitter background space. A first order Lagrangians looks now as follows:

$$\mathcal{L}_0 = \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b - \varepsilon^{\mu\nu\alpha} f_\mu{}^a D_\nu h_\alpha{}^a + \frac{\kappa}{2}\left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \tag{6}$$

and is invariant under the following local gauge transformations:

$$\delta h_{\mu a} = D_\mu \xi_a + \varepsilon_{\mu ab} \eta^b \qquad \delta f_\mu{}^a = D_\mu \eta_a + \kappa \varepsilon_{\mu ab} \xi^b \tag{7}$$

Working with the first order formalism it is very convenient to use tetrad formulation of the underlying (Anti) de Sitter space. We denote tetrad as $e_\mu{}^a$ (let us stress that it is not a dynamical quantity here, just a background field) and Lorentz covariant derivative as D_μ. (Anti) de Sitter space is a constant curvature space with zero torsion, so we have:

$$D_{[\mu} e_{\nu]}{}^a = 0, \qquad [D_\mu, D_\nu] v^a = \kappa (e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a) v_b \tag{8}$$

where $\kappa = -2\Lambda/(d-1)(d-2)$.

Now we switch on usual gravitational interaction by adding to the Lagrangian the following cubic terms:

$$\mathcal{L}_1 = -\frac{k}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} f_\mu{}^a f_\nu{}^b h_\alpha{}^c - \frac{k\kappa}{6} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{9}$$

as well as appropriate corrections to gauge transformation laws:

$$\delta_1 h_\mu{}^a = k \varepsilon^{abc} (f_\mu{}^b \xi^c + h_\mu{}^b \eta^c) \qquad \delta_1 f_\mu{}^a = k \varepsilon^{abc} (f_\mu{}^b \eta^c + \kappa h_\mu{}^b \xi^c) \tag{10}$$

Note that in $d = 3$ case this gives us complete interacting theory. Then we switch back to the usual variable: $f_\mu{}^a = \frac{1}{2} \varepsilon^{abc} \omega_\mu{}^{bc}$. Also in order to have canonical normalization of fields in dual theory (where ω is main dynamical quantity now, while h — just auxiliary field) we make a rescaling : $\omega \to \sqrt{\kappa}\omega$ and $h \to \frac{1}{\sqrt{\kappa}} h$. In this a quadratic Lagrangian takes the form:

$$\mathcal{L}_0 = \frac{\kappa}{2} \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac} \omega_\nu{}^{bc} - \frac{1}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\mu{}^{ab} D_\nu h_\alpha{}^c + \frac{1}{2} \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \tag{11}$$

and gauge transformations leaving it invariant (now $\eta^a = \frac{1}{2} \varepsilon^{abc} \eta_{bc}$)

$$\delta h_{\mu a} = D_\mu \xi_a + \kappa \eta_{\mu a} \qquad \delta \omega_\mu{}^{ab} = D_\mu \eta^{ab} - e_\mu{}^a \xi^b + e_\mu{}^b \xi^a \tag{12}$$

At the same time an interacting Lagrangian in these variables looks like:

$$\mathcal{L}_1 = -\frac{k\sqrt{\kappa}}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\mu{}^{ad} \omega_\nu{}^{bd} h_\alpha{}^c - \frac{k}{6\sqrt{\kappa}} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{13}$$

with appropriate corrections for gauge transformations:

$$\begin{aligned} \delta_1 h_\mu{}^a &= -k\sqrt{\kappa} (\omega_\mu{}^{ab} \xi^b + h_{\mu b} \eta^{ba}) \\ \delta_1 \omega_\mu{}^{ab} &= -k\sqrt{\kappa} (\omega_\mu{}^{ac} \eta^{cb} - \omega_\mu{}^{bc} \eta^{ca}) + \frac{k}{\sqrt{\kappa}} (h_\mu{}^a \xi^b - h_\mu{}^b \xi^a) \end{aligned} \tag{14}$$

Usually in passing to the second order formulation one solves algebraic equation of motion for the ω field (which geometrically give zero torsion condition). Then putting results back into the initial first order Lagrangian one obtains ordinary second order formulation in terms

of (symmetric) tensor field. Here we proceed another way and try to solve equation for h field which is also algebraic in (Anti) de Sitter background. This equation looks as:

$$\frac{\delta\mathcal{L}}{\delta h_\mu{}^a} = -\frac{1}{4}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} R_{\nu\alpha}{}^{bc} + \left\{ {}^{\mu\nu}_{ab} \right\} h_\nu{}^b - \frac{k}{2\sqrt{\kappa}}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\nu{}^b h_\alpha{}^c - \frac{k\sqrt{\kappa}}{2}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\nu{}^{bd}\omega_\alpha{}^{cd} \tag{15}$$

where now $R_{\mu\nu}{}^{ab} = D_\mu\omega_\nu{}^{ab} - D_\nu\omega_\mu{}^{ab}$. In the lowest order approximation we get:

$$h_\mu^{(1)a} = \hat{R}_\mu{}^a, \qquad \hat{R}_\mu{}^a = R_\mu{}^a - \frac{1}{4}e_\mu{}^a R \tag{16}$$

while a second order quadratic Lagrangian takes the form:

$$\mathcal{L}_0 = -\frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \hat{R}_\mu{}^a\hat{R}_\nu{}^b + \frac{\kappa}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} \tag{17}$$

Note, that appearance of quadratic mass-like terms is natural in (Anti) de Sitter background and does not mean that ω field becomes massive. It is important that besides usual gauge transformations $\delta\omega_\mu{}^{ab} = D_\mu\eta^{ab}$ this Lagrangian also invariant under the local shifts $\delta\omega_\mu{}^{ab} = -e_\mu{}^a\xi^b + e_\mu{}^b\xi^a$ which is a remnant of ξ-invariance of initial first order Lagrangian. To check this invariance one can use that under these transformations we have $\delta\hat{R}_\mu{}^a = D_\mu\xi^a$.

Now we proceed and consider next approximation with cubic interaction terms in the Lagrangian and linear terms in gauge transformation laws. Before we give explicit formulas let us discuss what kind of theory we obtain. Schematically the solution of h equation and cubic Lagrangian look like:

$$\begin{aligned} h^{(2)} &\sim \frac{k}{\sqrt{\kappa}}(D\omega)(D\omega) + k\sqrt{\kappa}\omega\omega \\ \mathcal{L}_1 &\sim \frac{k}{\sqrt{\kappa}}(D\omega)(D\omega)(D\omega) + k\sqrt{\kappa}(D\omega)\omega\omega \end{aligned} \tag{18}$$

So the "main" interaction terms are cubic three derivatives ones constructed from the gauge invariant field strengths $(D\omega)$, the coupling constant being $K = \frac{k}{\sqrt{\kappa}}$ and at this level theory is essentially abelian. Only the presence of nonzero cosmological term adds one derivative Yang-Mills type coupling with dimensionless coupling constant being $g = k\sqrt{\kappa}$. In this, our theory becomes non-abelian, the gauge group being the Lorentz group. The non-trivial interactions given above could be reproduced now in a kind of "flat" limit when $k \to 0$ and $\kappa \to 0$ keeping K fixed. Indeed, in this limit we obtain:

$$h_\mu^{(2)a} = K\left[-\hat{R}_\mu{}^b\hat{R}_b{}^a + \hat{R}_\mu{}^a\hat{R} + \frac{1}{4}e_\mu{}^a\hat{R}_b{}^c\hat{R}_c{}^b - \frac{1}{4}e_\mu{}^a\hat{R}^2\right] \tag{19}$$

while the cubic terms in the Lagrangian take the same simple form as before:

$$\mathcal{L}_1 = -\frac{K}{6}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{R}_\mu{}^a\hat{R}_\nu{}^b\hat{R}_\alpha{}^c \tag{20}$$

Besides the trivial at these limit invariance under the η^{ab} gauge transformations this Lagrangian is also invariant under the local shifts ξ^a with appropriate corrections:

$$\delta_1\omega_\mu{}^{ab} = K(\hat{R}_\mu{}^a\xi^b - \hat{R}_\mu{}^b\xi^a)$$

Let us stress that it is the invariance under these shifts that fixes the particular structure of cubic interactions among many other possible gauge invariant terms that could be easily constructed.

2 Dual gravity in $d \geq 4$

In this section we consider straightforward generalization of the procedure given above to the case of arbitrary $d \geq 4$ space-times. Again we start with the first order formulation of massless spin-2 particle in (Anti) de Sitter background with the Lagrangian:

$$\mathcal{L}_0 = \frac{\kappa}{2} \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac} \omega_\nu{}^{bc} - \frac{1}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} D_\mu \omega_\nu{}^{ab} h_\alpha{}^c + \frac{d-2}{2} \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \tag{21}$$

Here we have already made a rescaling of fields appropriate for dual version. Then we add the usual gravitational interactions at the first non-trivial (cubic) order:

$$\mathcal{L}_1 = \frac{k\sqrt{\kappa}}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\mu{}^{ad} \omega_\nu{}^{bd} h_\alpha{}^c - \frac{k}{4\sqrt{\kappa}} \left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} D_\mu \omega_\nu{}^{ab} h_\alpha{}^c h_\beta{}^d + \frac{(2d-5)k}{6\sqrt{\kappa}} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{22}$$

As is well known working with tetrad formulation of gravity and especially with supergravity theories it is very convenient to use the so called "1 and 1/2" order formalism. But here to construct a dual theory we have to work in a "honest" first order formalism taking into account gauge transformations for all fields. In this approximation they have the following form:

$$\begin{aligned}
\delta_1 \omega_\mu{}^{ab} &= \frac{k}{\sqrt{\kappa}} [\xi^\nu R_{\nu\mu}{}^{ab} + (R_\mu{}^a \xi^b - R_\mu{}^b \xi^a) + \frac{1}{d-2} \xi^\nu (e_\mu{}^a R_\nu{}^b - e_\mu{}^b R_\nu{}^a) - \\
&\quad - \frac{1}{2(d-2)} (e_\mu{}^a \xi^b - e_\mu{}^b \xi^a) R - (d-2)(h_\mu{}^a \xi^b - h_\mu{}^b \xi^a)] \\
\delta_1 h_\mu{}^a &= k\sqrt{\kappa} \omega_\mu{}^{ab} \xi^b
\end{aligned} \tag{23}$$

for the ξ^a-transformations as well as

$$\delta_1 \omega_\mu{}^{ab} = k\sqrt{\kappa} (\omega_\mu{}^{ac} \eta^{cb} - \omega_\mu{}^{bc} \eta^{ca}) \qquad \delta_1 h_\mu{}^a = k\sqrt{\kappa} h_{\mu b} \eta^{ba} \tag{24}$$

for the η^{ab}-ones. Note that the main difference from the $d = 3$ case is rather complicated form for the ξ-transformations of ω field. As we will see this leads to the essential difference in the structure of interacting Lagrangian. Now we try to solve algebraic equation for h field which in this approximation looks as follows:

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta h_\mu{}^a} &= -\frac{1}{4} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} R_{\nu\alpha}{}^{bc} + (d-2) \left\{ {}^{\mu\nu}_{ab} \right\} h_\nu{}^b - \frac{k}{4\sqrt{\kappa}} \left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} R_{\nu\alpha}{}^{bc} h_\beta{}^d + \\
&\quad + \frac{(2d-5)k}{2\sqrt{\kappa}} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\nu{}^b h_\alpha{}^c + \frac{k\sqrt{\kappa}}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\nu{}^{bd} \omega_\alpha{}^{cd}
\end{aligned} \tag{25}$$

This equation is a non-linear one. Moreover, if one consider next to the linear approximations then one obtains even more non-linear terms. So it seems hardly possible to get general solution of this equation, but nothing prevent us from solving it iteratively, order by order. Here we restrict ourselves by the linear approximation as in the previous case. In the lowest order approximation we get:

$$h_\mu^{(1)a} = \frac{1}{d-2} \hat{R}_\mu{}^a, \qquad \hat{R}_\mu{}^a = R_\mu{}^a - \frac{1}{2(d-1)} e_\mu{}^a R \tag{26}$$

and in this notations the structure of quadratic second derivative Lagrangian looks very similar to the $d = 3$ case:

$$\mathcal{L}_0 = -\frac{1}{2(d-2)} \left\{ {}^{\mu\nu}_{ab} \right\} \hat{R}_\mu{}^a \hat{R}_\nu{}^b + \frac{\kappa}{2} \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac} \omega_\nu{}^{bc} \tag{27}$$

The formulas in the next approximation could be greatly simplified if we introduce traceless conformal Weyl tensor:

$$C_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - \frac{1}{d-2} e_{[\mu}{}^{[a} R_{\nu]}{}^{b]} + \frac{1}{(d-1)(d-2)} e_\mu{}^{[a} e_\nu{}^{b]} R \tag{28}$$

in this, the following useful relation holds:

$$R_{\mu\nu}{}^{ab} = C_{\mu\nu}{}^{ab} + \frac{1}{d-2} e_{[\mu}{}^{[a} \hat{R}_{\nu]}{}^{b]} \tag{29}$$

As in the $d = 3$ case it is possible to consider a "flat" limit with when $k \to 0$ and $\kappa \to 0$ keeping $K = \frac{k}{\sqrt{\kappa}}$ fixed. In this limit a solution of h equation in the next order gives:

$$h_\mu^{(2)a} = -\frac{K}{(d-2)^3} [(d-2) C_{\mu\nu}{}^{ab} \hat{R}_b{}^\nu - \hat{R}_\mu{}^\nu \hat{R}_\nu{}^a + \hat{R}_\mu{}^a \hat{R} + \frac{1}{2(d-1)} e_\mu{}^a [(\hat{R}\hat{R}) - \hat{R}^2]] \tag{30}$$

Then putting this expression back to the initial first order Lagrangian and keeping only cubic terms we obtain the following three derivatives Lagrangian:

$$\mathcal{L}_1 = -\frac{K}{2(d-2)^2} [\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} C_{\mu\nu}{}^{ab} \hat{R}_\alpha{}^c \hat{R}_\beta{}^d + \frac{d-4}{3(d-2)} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{R}_\mu{}^a \hat{R}_\nu{}^b \hat{R}_\alpha{}^c] \tag{31}$$

Again this particular structure of the Lagrangian is fixed not only by the invariance under the usual gauge transformations $\delta\omega_\mu{}^{ab} = \partial_\mu \eta^{ab}$, but also by the invariance under the local ξ shifts with the linear terms being:

$$\delta_1 \omega_\mu{}^{ab} = K[\xi^\nu C_{\nu\mu}{}^{ab} + \frac{1}{d-2} (\hat{R}_\mu{}^b \xi^a - \hat{R}_\mu{}^a \xi^b)] \tag{32}$$

The following identities turn out to be useful:

$$\begin{aligned} D_a C_{\mu\nu}{}^{ab} &= \frac{1}{d-2} (D_\mu \hat{R}_\nu{}^b - D_\nu \hat{R}_\mu{}^b) \\ D_a \hat{R}_\mu{}^a &= D_\mu \hat{R} \end{aligned} \tag{33}$$

Note, that the general structure of the Lagrangian obtained is in agreement with the $d = 3$ case. Indeed, in $d = 3$ conformal Weyl tensor is identically zero, so the first term is absent. It is interesting to note that the $d = 4$ case is also special, because in this and only this case the second term is absent.

Conclusion

In this paper we have shown that there indeed exists a dual formulation of gravity in terms of Lorentz connection $\omega_\mu{}^{ab}$ field. Such formulation turns out to be highly non-linear higher derivatives theory, so it is not an easy task (if at all possible) to give compact formulation at full non-linear level. However it is possible to construct such theory iteratively, order by order in fields as we have done in the linear approximation here. Also we have shown that the so called exotic parity-violating interactions for massless spin-2 particles could be considered just as such dual formulation of usual gravitational interactions.

References

[1] A. S. Matveev, M. A. Vasiliev, *"On Dual Formulation for Higher Spin Gauge Fields in (A)dS"*, Phys. Lett. **B609** (2005) 157, hep-th/0410249.

[2] Yu. M. Zinoviev *"On Dual Formulations for Massive Tensor Fields"*, hep-th/0504081.

[3] N. Boulanger, S. Cnockaert, M. Henneaux, *"A Note on Spin S Duality"*, JHEP **0306** (2003) 060, hep-th/0306023.

[4] M. Contreras, J. Zanelli, *"A note on the spin connection representation of gravity"*, Class.Quant.Grav. **16** (1999) 2125, hep-th/9904022.

[5] B. Julia, *"Towards dual gravity"*, talk at the International Conference on Theoretical Physics, April 11-16, Moscow.

[6] N. Boulanger, L. Gualteri, *"An exotic theory of massless spin-two fields in three dimensions"*, Class. Quant. Grav. **18** 1485, hep-th/0012003.

[7] S. C. Anco *"Parity violating spin-two gauge theories"*, Phys. Rev. **D67** (2003) 124007, gr-qc/0305026.

[8] Yu. M. Zinoviev *"First Order Formalism for Mixed Symmetry Tensor Fields"*, hep-th/0304067.